

# Single Photon Ignition of Two-photon Super-fluorescence through the Vacuum of Electromagnetic Field

Nicolae A. Enaki

Academiei str.5, Institute of Applied Physics of Academy of sciences of Moldova,  
Chisinau, MD 2028, Republic of Moldova

December 8, 2010

## Abstract

The ignition of two-quantum collective emission of inverted sub-ensemble of radiators due to mutual interaction of this sub-ensemble with other two dipole active atomic subsystems in process of two-photon exchanges between the atoms through the vacuum field is proposed. The three particle resonances between two-photon and single quantum transitions of inverted radiators from the ensemble are proposed for acceleration of collective decay rate of bi-photons, obtained relatively dipole-forbidden transitions of excited atomic sub-ensemble. This mutual interaction between three super-fluorescent processes in subatomic ensembles take place relatively dipole-forbidden transitions in one of radiator subsystem. The collective resonance emission and absorption of two-quanta have nontraditional behavior, accompanied with acceleration and inhibition of collective emission processes of photons.

## 1 Introduction

A great deal of attention is currently devoted to the problem of coherence which appears not only between the quanta but between groups of quanta too. The generation of non-classical coherent electromagnetic field in multi-photon emission and the interaction of coherent radiation with matter (nuclei, atoms and solids) have been subjects of a number of theoretical and experimental studies in recent years [1]-[3]. Examples include the higher-order coherence in multi-photon generation of light the two-photon micro-maser emission [3], two-photon lasers the parametric down conversion, four-wave mixing and other effects in optical diapason [2], and the possibility of coherent generation of photons in  $x$ -ray and  $\gamma$ -ray spectral regions.

In this article it is proposed to investigate the cooperative two-photon emission from inverted system of radiators stimulated by single photon super-fluorescent pulses in two-quantum resonance with dipole forbidden atomic transition. Since the two-photon cooperative phenomenon has the small two-photon cooperative emission time [9], we propose to extend our attention to the new tape of cooperative resonance interaction between three radiators in which single photon transitions of two radiators which enter in two-photon resonance with dipole forbidden transition of third atom. This cooperative three

particle interaction take place through the vacuum fluctuations of electromagnetic field and can amplify or diminish the spontaneous emission rates of the atoms. In order to obtain more powerful pulses of entangled photons it is proposed the cooperative interaction between three atomic subsystems in which one of them are inverted relatively dipole forbidden transition  $|2S\rangle \rightarrow |S\rangle$  of Hydrogen like or Helium Like atoms [4]-[8]. Taking in to account the elementary acts of two photon interaction between radiators we archived the improvement of two-photon emission rate of the system of radiators in comparison with two-photon super-fluorescence [9]. In this article it is examined the mutual influence of two single-photon super-fluorescence processes and two-photon cooperative emission of the atomic system relatively dipole forbidden transition. The phenomenon of new cooperative emission takes in to account the three particle mutual interaction with vacuum of electromagnetic field in which the product of vacuum polarization of two atoms enter in to resonance with two-photon polarization of dipole forbidden transition of Hydrogen-like or Helium-like radiator. It has been shown that in the process of spontaneous radiation, the radiators (nuclei, atoms) enter a regime of single and two-photon super-radiance and the rate of photon pair (bi-photon) emission increases (or decreases) due to new three particle cooperative phenomenon, which appear between single and two-photon spontaneous emission subgroups of radiators. It has been demonstrated, that for hydrogen-like and helium-like atoms [10] the dipole-forbidden transitions can generate more powerful pulse of entangled photon pairs (bi-photons) under the influence of single photon super-radiance.

It is important to note that for coherent radiation of such system, was studied for the dimension of a radiating system smaller than the radiation wavelength. It is, however, interesting to study this type of cooperative emission between three radiator subsystems in extended system of radiators. The possibilities of two-photon cooperative resonances between three radiators replaced at distance larger than emission wavelength are studied too. I emphasize here, that the problem of cooperation between two single photon cooperative emission subsystems and one two-photon cooperative emission subsystem is more complicated than the similar problem of single [11] or two-photon [9] super-radiances in extended system. In Dicke's super-radiance, the exchange integral between  $j$ -th and  $l$ -th atoms is described by more simple exchange integral proportional to  $\sin[k_0 r_{jl}]/(k_0 r_{jl})$  while in two-photon super-radiance by more complicated function  $\sin[(2k_0 - k)r_{jl}]/[(2k_0 - k)r_{jl}] \times \sin[kr_{jl}]/(kr_{jl})$ , where  $r_{jl}$  and  $k_i$  are the distance between the radiators and wave vector of emitted photons respectively. The three particle exchange integral between  $j$ -th,  $m$ -th and  $l$ -th atoms was obtained in this paper taking in to account the two-quantum exchanges between two radiators proposed in papers [9], [12]. The more complicated exchange integrals between two radiators with dipole active transition and one radiator with dipole forbidden transition is given in Appendix of this paper.

## 2 Interaction Hamiltonian and Master Equation

Let us consider the interaction of three subsystems of radiators  $R$ ,  $S$ , and  $D$  thorough vacuum of electromagnetic field. The first two groups,  $R$  and  $S$ , are prepared in excited state  $|e_r\rangle \otimes |e_s\rangle$  and can pass in to Dicke super-radiance regime [11] relatively the dipole active transitions  $e_r \rightarrow g_r$  and  $e_s \rightarrow g_s$  at frequencies  $\omega_r$  and  $\omega_s$  (see figure 1). The  $D$  atomic subsystem is prepared in excited state  $|e_d\rangle$  and

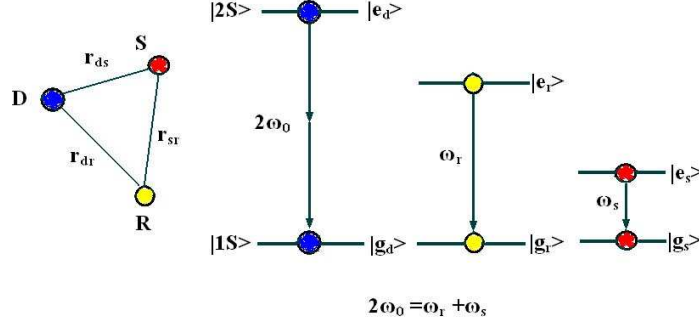


Figure 1: The resonance between two-photon transitions of  $D$  atomic subsystem and two dipole active atomic subsystems  $R$  and  $S$ . As an example is represented three atoms  $D$ ,  $R$  and  $S$  situated at relatively distances  $r_{ds}$ ,  $r_{dr}$  and  $r_{rs}$ . One of condition of exchange energies between the subsystems is the resonance between two-photon and single photon transitions  $2\omega_0 = \omega_r + \omega_s$ .

relatively dipole forbidden transition  $e_d \rightarrow g_d$  and can pass in the ground state  $|g_d\rangle$  simultaneously generation two quanta [9]. Let us consider the simple cooperative stimulation of two-photon emission of  $D$  system stimulated by  $R$  and  $S$  radiator subsystems.

In this case it is established the resonance between the dipole active atomic subgroups  $R$ ,  $S$  and dipole forbidden radiators of  $D$  ensemble and the cooperative stimulation of two-quantum collective transition is possible. Indeed, considering that the conservation energy law is established between these groups,  $\hbar(\omega_r + \omega_s) = 2\hbar\omega_0$ , one can proposed the following Hamiltonian of interaction of radiators with electromagnetic field

$$\begin{aligned}
H &= H_0 + \lambda H_I; \\
H_0 &= \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_{j=1}^{N_r} \hbar\omega_r R_{zj} + \sum_{l=1}^{N_s} \hbar\omega_s S_{zl} + 2 \sum_{m=1}^N \hbar\omega_0 D_{zm}; \\
\lambda H_I &= - \sum_k \sum_{j=1}^{N_a} (\mathbf{d}_r, \mathbf{g}_k) \{ R_j^+ a_k \exp[i(\mathbf{k}, \mathbf{r}_j)] + R_j^- a_k^\dagger \exp[-i(\mathbf{k}, \mathbf{r}_j)] \} \\
&\quad - \sum_k \sum_{l=1}^{N_b} (\mathbf{d}_s, \mathbf{g}_k) \{ S_l^+ a_k \exp[i(\mathbf{k}, \mathbf{r}_l)] + S_l^- a_k^\dagger \exp[-i(\mathbf{k}, \mathbf{r}_l)] \} \\
&\quad - \sum_{k_1, k_2} \sum_{m=1}^{N_b} (\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1}) (\mathbf{n}_{ei}, \mathbf{e}_{\lambda_2}) q(\omega_1, \omega_2) \\
&\quad \times \{ D_m^+ a_{k_2} a_{k_1} \exp[i(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{r}_m)] \\
&\quad + D_m^- a_{k_1}^\dagger a_{k_2}^\dagger \exp[-i(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{r}_l)] \}.
\end{aligned} \tag{1}$$

Here

$$q(\omega_1, \omega_2) = \frac{d_{23} d_{31} g_{k_2} g_{k_1}}{2\hbar} \left\{ \frac{1}{\omega_{32} + \omega_{k_1}} + \frac{1}{\omega_{31} - \omega_{k_2}} \right\}, \quad \mathbf{g}_k = \sqrt{\frac{2\pi\hbar\omega_k}{V}} \boldsymbol{\epsilon}_\lambda,$$

$a_k$  and  $a_k^\dagger$  are annihilation and creation operators of EMF photons with wave vector  $\mathbf{k}$ , polarization  $\boldsymbol{\epsilon}_\lambda$

and frequency  $\omega_k$ ;  $\mathbf{d}_r$  and  $\mathbf{d}_s$  are dipole momentum transition between the ground and excited states for  $R$  and  $S$  atomic subsystems;  $d_{ei}$  and  $d_{eg}$  are dipole momentum transitions in the three level system of atomic group  $D$ . The operators of  $R$ ,  $S$ , and  $D$  atomic subsystems satisfy the commutation relations for  $SU(2)$  algebra  $[J^+, J^-] = 2J_z$ ;  $[J_z, J^\pm] = \pm J^\pm$ , where  $J^\pm$  is equivalent with  $R^\pm$ ,  $S^\pm$  and  $D^\pm$ . Invertin operator  $J_z$  is consider similar to  $R_z$ ,  $S_z$  and  $D_z$  respectively. The operators of electromagnetic field satisfy the commutation relation  $[a_k, a_{k'}^\dagger] = \delta_{k,k'}$ ;  $[a_k^\dagger, a_{k'}^\dagger] = 0$ , where  $k = (\mathbf{k}, \lambda)$  is the wave vector and polarization of the photon. Taking in to account the Hamiltonian 1, let us represent the solution of Haisenberg equation through the sources and free part operators

$$a_k(t) = a_k(0) \exp[-i\omega_k t] + a_{ks}(t), \quad (2)$$

where the source part is

$$\begin{aligned} a_{ks}(t) = & \frac{i(\mathbf{d}_a, \mathbf{g}_k)}{\hbar} \sum_{l=1}^{N_a} \exp[-i(\mathbf{k}, \mathbf{r}_l)] \int_0^t d\tau \exp[-i\omega_k \tau] R_l^-(t - \tau) \\ & + i \frac{(\mathbf{d}_b, \mathbf{g}_k)}{\hbar} \sum_{j=1}^{N_b} \exp[-i(\mathbf{k}, \mathbf{r}_j)] \int_0^t d\tau \exp[-i\omega_k \tau] S_j^-(t - \tau) \\ & + 2i \sum_{n=1}^{N_b} \sum_{k_1} \frac{(\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1})(\mathbf{n}_{ei}, \mathbf{e}_\lambda) q(\omega_1, \omega)}{\hbar} \exp[-i(\mathbf{k}_1 + \mathbf{k}, \mathbf{r}_n)] \\ & \times \int_0^t d\tau \exp[-i\omega_k \tau] D_n^-(t - \tau) a_{k_1}^\dagger(t - \tau); \quad [a_{ks}^\dagger(t) = [a_s(t)]^+]. \end{aligned}$$

Taking in to account that  $a_k(0)|0\rangle_{ph} = \langle 0|_{ph} a_k^\dagger(0) = 0$  we can partially eliminate the EMF field operators from the mean value of Hesenberg equation for arbitrary atomic operator  $O(t)$

$$\begin{aligned} \frac{d}{dt} \langle O(t) \rangle = & -i \sum_k \sum_{j=1}^{N_a} \frac{(\mathbf{d}_a, \mathbf{g}_k)}{\hbar} \langle [R_j^+(t), O(t)] a_{ks}(t) \rangle \exp[i(\mathbf{k}, \mathbf{r}_j)] \\ & -i \sum_k \sum_{l=1}^{N_b} \frac{(\mathbf{d}_b, \mathbf{g}_k)}{\hbar} \langle [S_l^+(t), O(t)] a_{ks}(t) \rangle \exp[i(\mathbf{k}, \mathbf{r}_l)] \\ & -i \sum_{k_1, k} \sum_{m=1}^{N_b} \frac{(\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1}(k_1))(\mathbf{n}_{ei}, \mathbf{e}_\lambda(k)) q(\omega_{k_1}, \omega_k)}{\hbar} \\ & \times \langle [D_m^+(t), O(t)] a_{k_1}(t) a_{ks}(t) \rangle \exp[i(\mathbf{k}_1 + \mathbf{k}, \mathbf{r}_m)] + H.C.(O^+ \rightarrow O). \end{aligned} \quad (3)$$

Here the mean values of Hesenberg operators are considered taking into account the initial state of the system  $|\Psi_r(0)\rangle \otimes |0\rangle_{ph}$ , where  $|\Psi_r(0)\rangle$  is the state of radiator subsystem, and  $|0\rangle_{ph}$  is the vacuum state of EMF. We are interested in the total elimination of operators of electromagnetic field from the expression (3) For elimination of operators of electromagnetic field we formulate the lemma

**Lemma 1** *If Bose  $a_k(t)$  and  $a_k^+(t)$  operators lie between the two operators of the atomic subsystem  $A(t_1)$  and  $B(t_2)$  ( $A(t_1)$ ,  $B(t_2)$  don't contain the operators  $a_k$  and  $a_k$ ) belonging to other times, the elimination of the free part of these operators yields the following expression for the correlation:*

$$\langle A(t_1) a_k(t) B(t_2) \rangle = \langle A(t_1) a_{ks}(t) B(t_2) \rangle$$

$$\begin{aligned}
& - e^{-i\omega_3(t-t_2)} \langle A(t_1)[a_{ks}(t_2), B(t_2)] \rangle, \\
\langle A(t_1)a_k^+(t)B(t_2) \rangle &= \langle A(t_1)a_{ks}^+(t)B(t_2) \rangle \\
& - e^{i\omega_3(t-t_1)} \langle [A(t_1), a_{ks}^+(t_1)]B(t_2) \rangle.
\end{aligned} \tag{4}$$

**Proof.** The commutations in (4) play the highest role in the two-photon spontaneous emission and only such commutations bring the main contribution to the two-photon process. The problem is reduced to the elimination of vacuum part lies between the operators  $A(t_1)$  and  $B(t_2)$

$$< A(t_1)a_k(t)B(t_2) > = < A(t_1)(a_k^v(t) + a_{ks}(t))B(t_2) > \tag{5}$$

Since  $a_k(t) = a_k^v(t) + a_{ks}(t)$ , we will represent the vacuum part  $a_k^v(t) = a_k(0) \exp[-i\omega_k t]$  through the vacuum-operator at time  $t_1$  and taking in to account the identity (2) we can represent the vacuum part in the following form  $a_k^v(t) = a_k^v(t_2)e^{-i\omega_k(t-t_2)} = \{a_k(t_2) - a_{ks}(t_2)\}e^{-i\omega_k(t-t_2)}$ . After substitution of  $a_k^v(t)$  into the correlation it is obtain

$$\begin{aligned}
\langle A(t_1)a_k(t)B(t_2) \rangle &= \langle A(t_1)a_{ks}(t)B(t_2) \rangle \\
&+ e^{-i\omega_k(t-t_2)} \langle A(t_1)\{a_k(t_2) - a_{ks}(t_2)\}B(t_2) \rangle,
\end{aligned}$$

We observe that  $a_k(t_2)$  commutes with the operator  $B(t_2)$ . Consequently taking into account that

$a_k(t_2) B(t_2)|0 \rangle = B(t_2)a_{ks}(t_2)|0 \rangle$ , it is easily obtain that

$$\langle A(t_1)\{a_k(t_2) - a_{ks}(t_2)\} B(t_2) \rangle = -\langle B(t_2)[a_{ks}(t_2), B(t_2)] \rangle.$$

This relation proofs the Lemma. ■

This lemma (4) can be used in the last term of generalized equation (3) for correlation functions  $\langle [D_j^+(t), O(t)]a_k(t)S_n^-(t-\tau) \rangle$ ,  $\langle [D_j^+(t), O(t)]a_k(t)R_l^-(t-\tau) \rangle$  and  $\langle [R_j^+(t), O(t)]D_n^-(t-\tau)a_{k_1}^\dagger(t-\tau) \rangle$ ,  $\langle [S_l^+(t), O(t)]D_n^-(t-\tau)a_{k_1}^\dagger(t-\tau) \rangle$ . Indeed taking in to account the lemma (4) the above correlation functions can be represented through atomic operators

$$\begin{aligned}
\langle [D_j^+(t), O(t)]a_k(t)S_n^-(t-\tau) \rangle &= \langle [D_j^+(t), O(t)]a_{ks}(t)S_n^-(t-\tau) \rangle \\
&- e^{-i\omega_k\tau} \langle [D_j^+(t), O(t)][a_{ks}(t-\tau), S_n^-(t-\tau)] \rangle,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\langle [R_j^+(t), O(t)]a_{k_1}^\dagger(t-\tau)D_n^-(t-\tau) \rangle &= \langle [R_j^+(t), O(t)]a_{ks}^\dagger(t-\tau)D_n^-(t-\tau) \rangle \\
&- e^{-i\omega_k\tau} \langle [[R_j^+(t), O(t)], a_{ks}^\dagger(t)]D_n^-(t-\tau) \rangle.
\end{aligned} \tag{7}$$

The interaction between the atomic subsystems can be found in the third order of interaction constants with the subsystems  $S$ ,  $R$  and  $D$  respectively  $(\mathbf{d}_r, \mathbf{g}_k)(\mathbf{d}_s, \mathbf{g}_k)q(\omega_1, \omega_2)$  According with this condition the smooth correlation functions is obtained only for the following terms of expressions (6) and (7) :  $\langle R_l^-(t-\tau')[D_j^+(t), O(t)]S_n^-(t-\tau) \rangle$  and  $\langle R_l^+(t-\tau')[R_j^+(t), O(t)]D_n^-(t-\tau) \rangle$ . The contribution of other terms of the expressions (6) and (7) give the contribution more hair order on the decomposition on the small parameter  $\lambda$  of the interaction Hamiltonian (1).

The lemma (4) is non-applicable for correlation functions in which it is meet simultaneously the creation and annihilations Boson operators belonging to different time intervals:  $\langle [D_j^+(t), O(t)]a_k(t)D_n^-(t-\tau)a_{k_1}^\dagger(t-\tau) \rangle$  and its hermit conjugate part  $\langle a_{k_1}(t-\tau)D_n^+(t-\tau)a_k^\dagger(t)[O(t), D_j^-(t)] \rangle$ . In order to eliminate the vacuum part of operators  $a_k(t)$  and  $a_k^\dagger(t')$  let us formulate the following rule.

**Lemma 2** *If the operators  $A(t_1)$  contains the creation operators of EMF and  $B(t_2)$  contains the annihilation operators of EMF the elimination of vacuum part of annihilation  $a_k(t)$  or creation  $a_k^\dagger(t)$  operators situated between these operators  $A(t_1)$  and  $B(t_2)$  takes place according with Lemma 1. In opposite case, when operator  $A(t_1)$  can be represented through the product of atomic operator  $\mathcal{A}(t_1)$  and annihilation field operators  $A(t_1)=\mathcal{A}(t_1)a_{k_1}(t_1)a_{k_2}(t_1)...a_{k_n}(t_1)$  . the operator  $B(t_2)$  is represented through the product of creation field operators and atomic operator  $\mathcal{B}(t_2)$  so that  $B(t_2)=\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2)$  the elimination of vacuum part of the operators  $a_k(t)$  and  $a_k^\dagger(t)$  can be represented in the following form*

$$\begin{aligned} \langle A(t_1)a_k(t)B(t_2) \rangle &= \langle A(t_1)a_{ks}(t)B(t_2) \rangle \\ &- \exp[-i\omega_k(t-t_2)]\{\langle A(t_1)[a_{ks}(t_2), B(t_2)] \rangle \\ &- \delta_{k,k_1} \langle A(t_1)\mathcal{B}(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2) \rangle \\ &- \dots - \delta_{k,k_m} \langle A(t_1)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)...a_{k_{m-1}}^\dagger(t_2) \rangle\}, \end{aligned} \quad (8)$$

$$\begin{aligned} \langle A(t_1)a_k^\dagger(t)B(t_2) \rangle &= \langle A(t_1)a_{ks}^\dagger(t)B(t_2) \rangle \\ &- \exp[i\omega_k(t-t_2)]\{\langle [A(t_1)a_{ks}^\dagger(t_1)]B(t_2) \rangle \\ &- \delta_{k,k_1} \langle \mathcal{A}(t_1)a_{k_2}(t_1)...a_{k_n}(t_1)B(t_2) \rangle \\ &- \dots - \delta_{k,k_n} \langle \mathcal{A}(t_1)a_{k_1}(t_1)a_{k_2}(t_1)...a_{k_{n-1}}(t_1)B(t_2) \rangle\}. \end{aligned} \quad (9)$$

**Proof.** Taking in to account the lemma (4), we can represent the third correlation  $\langle A(t_1)a_k(t)B(t_2) \rangle$  of expression (8) in the following form

$$\begin{aligned} \langle A(t_1)a_k(t)B(t_2) \rangle &= \langle A(t_1)a_{ks}(t)B(t_2) \rangle \\ &+ \exp[-i\omega_k(t-t_2)] \langle A(t_1)(a_k(t_2) - a_{ks}(t_2))B(t_2) \rangle \\ &- \exp[-i\omega_k(t-t_2)]\{\langle A(t_1)[a_{ks}(t_2), B(t_2)] \rangle\}. \end{aligned} \quad (10)$$

According with explicit expression of operator,  $B(t_2) = \mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2)$ , let us introduced it in the third term of right hand site of expression (10). Following the commutation roles of boson operators of electromagnetic field, the operator  $a_k(t_2)$  can be permuted in the right hand site of the correlation. Taking in to consideration that  $(a_{ks}(t_2) + a_{kv}(t_2))|0\rangle = a_{ks}(t_2)|0\rangle$ , this term becomes

$$\begin{aligned} \langle A(t_1)a_k(t_2)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2) \rangle &= \\ \delta_{k,k_1} \langle A(t_1)\mathcal{B}(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2) \rangle & \\ + \dots + \delta_{k,k_m} \langle A(t_1)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)...a_{k_{m-1}}^\dagger(t_2) \rangle & \\ + \langle A(t_1)\mathcal{B}(t_2)a_{k_1}^\dagger(t_2)a_{k_2}^\dagger(t_2)...a_{k_m}^\dagger(t_2)a_{ks}(t_2) \rangle. & \end{aligned} \quad (11)$$

Introducing this relation in (11) it is not difficult to observe that the new expression for correlation  $\langle A(t_1)a_k(t)B(t_2) \rangle$  coincides with (8). The similar procedure of permutation of vacuum part of creation operator  $a_k^\dagger(t)$  demonstrates the identity (9) of Lemma 2. ■

According with lemma (8) we obtain the following expression for correlation function

$$\begin{aligned} & \langle [D_j^+(t), O(t)] a_k(t) D_n^-(t-\tau) a_{k_1}^\dagger(t-\tau) \rangle = \langle [D_j^+(t), O(t)] a_{ks}(t) D_n^-(t-\tau) a_{k_1}^\dagger(t-\tau) \rangle \\ & - \exp[-i\omega_k\tau] \{ \langle [D_j^+(t), O(t)] [a_{ks}(t-\tau), D_n^-(t-\tau) a_{k_1}^\dagger(t-\tau)] \rangle - \delta_{k,k_1} \langle [D_j^+(t), O(t)] D_n^-(t-\tau) \rangle \} \end{aligned} \quad (12)$$

The next step of elimination of operator  $a_{k_1}^\dagger(t-\tau)$  from this expression must be do taking in-to account the lemma (4). When the first and second order interaction constants have the same small magnitude  $\lambda \sim (\mathbf{d}_s, \mathbf{g}_k) \approx q(\omega_{k_1}, \omega_k)$ , in Born approximation we take in to account only the last term of expression (12). the interference contribution of which is proportional to  $\lambda^3$ . As follows from the representation (2) and (3) the procedure of elimination mast continue. Indeed introducing again this equation in the right hand cite of equation (14) we obtain the following master equation for arbitrary operator  $O(t)$  in thread approximation on the interaction constant  $\lambda$

$$\begin{aligned} \frac{d\langle O(t) \rangle}{dt} &= \sum_k \sum_{l,j=1}^{N_r} \frac{(\mathbf{d}_a, \mathbf{g}_k)^2}{\hbar^2} \int_0^t d\tau \exp[-i\omega_k\tau + i(\mathbf{k}, \mathbf{r}_j - \mathbf{r}_l)] \langle [R_j^+(t), O(t)] R_l^-(t-\tau) \rangle \\ &+ \sum_k \sum_{l,l=1}^{N_s} \frac{(\mathbf{d}_b, \mathbf{g}_k)^2}{\hbar^2} \int_0^t d\tau \exp[-i\omega_k\tau + i(\mathbf{k}, \mathbf{r}_j - \mathbf{r}_l)] \langle [S_j^+(t), O(t)] S_l^-(t-\tau) \rangle \\ &+ \sum_{k_1, k_2}^N \sum_{l,j=1}^N \frac{(\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1})^2 (\mathbf{n}_{ei}, \mathbf{e}_{\lambda_2})^2 q^2(\omega_1, \omega_2)}{\hbar^2} \int_0^t d\tau \langle [D_j^+(t), O(t)] D_l^-(t-\tau) \rangle \\ &\times \exp[-i(2\omega_0 - \omega_{k_1} - \omega_{k_2})\tau] \exp[i(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{r}_j - \mathbf{r}_l)] \\ &+ i \sum_{k,k'}^N \sum_{n=1}^N \sum_{j=1}^N \sum_{l=1}^N \frac{(\mathbf{n}_{eg}, \mathbf{e}_{\lambda}(\mathbf{k})) (\mathbf{n}_{ei}, \mathbf{e}_{\lambda}(\mathbf{k}')) q(\omega_k, \omega_{k'})}{\hbar^3} \int_0^t d\tau \int_0^t d\tau' \exp[i(\mathbf{k}, \mathbf{r}_n - \mathbf{r}_l) - i\omega_k\tau] \\ &\times \exp[i(\mathbf{k}', \mathbf{r}_n - \mathbf{r}_l) - i\omega_{k'}\tau'] [(\mathbf{d}_s, \mathbf{g}_{k'}) (\mathbf{d}_r, \mathbf{g}_k) \langle [D_n^+(t), O(t)] R_j^-(t-\tau) S_l^-(t-\tau') \rangle \\ &+ (\mathbf{d}_r, \mathbf{g}_{k'}) (\mathbf{d}_s, \mathbf{g}_k) \langle [D_n^+(t), O(t)] S_j^-(t-\tau) R_l^-(t-\tau') \rangle] \\ &- i \sum_{k,k'}^N \sum_{m=1}^N \sum_{j=1}^N \sum_{l=1}^N \frac{(\mathbf{n}_{eg}, \mathbf{e}_{\lambda}(\mathbf{k})) (\mathbf{n}_{ei}, \mathbf{e}_{\lambda}(\mathbf{k}')) q(\omega_k, \omega_{k'})}{\hbar^3} \int_0^t d\tau' \int_0^t d\tau \exp[i\omega_k\tau - i(\mathbf{k}, \mathbf{r}_m - \mathbf{r}_l)] \\ &\times \exp[-i\omega_k\tau' - i\omega_{k'}\tau' + i(\mathbf{k}', \mathbf{r}_j - \mathbf{r}_m)] [(\mathbf{d}_s, \mathbf{g}_k) (\mathbf{d}_r, \mathbf{g}_{k'}) \langle S_l^+(t-\tau) [R_j^+(t), O(t)] D_m^-(t-\tau') \rangle \\ &+ (\mathbf{d}_s, \mathbf{g}_k) (\mathbf{d}_r, \mathbf{g}_{k'}) \langle R_l^+(t-\tau) [S_j^+(t), O(t)] D_m^-(t-\tau') \rangle] + H.c.(O^+ \rightarrow O). \end{aligned} \quad (13)$$

The traditional Born-Marcov approximation in the right hand site of equation ( 13) give us the divergent functions. In order to understood this we approximate the right hand site of equation (2) with following expression

$$\begin{aligned} a_{ks}(t) &= \frac{(\mathbf{d}_a, \mathbf{g}_k)}{\hbar} \sum_{l=1}^{N_a} R_l^-(t) \exp[-i(\mathbf{k}, \mathbf{r}_l) \zeta^*(\omega_k - \omega_a)] \\ &+ \frac{(\mathbf{d}_b, \mathbf{g}_k)}{\hbar} \sum_{j=1}^{N_b} S_j^-(t) \exp[-i(\mathbf{k}, \mathbf{r}_j) \zeta^*(\omega_k - \omega_b)] \\ &+ 2 \sum_{n=1}^{N_b} \sum_{k_1} \frac{(\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1}) (\mathbf{n}_{ei}, \mathbf{e}_{\lambda}) q(\omega_1, \omega)}{\hbar} \\ &\times D_n^-(t) a_{k_1}^\dagger(t) \exp[-i(\mathbf{k}_1 + \mathbf{k}, \mathbf{r}_n) \zeta^*(\omega_k + \omega_{k_1} - 2\omega_0)], \end{aligned} \quad (14)$$

in the Born-Marcovian approximation [15] , [13]. The small parameter in this approximation is the ratio of retardation time to cooperative spontaneous emission times of the subsystem,  $\tau/\tau_i \ll 1$ . Here  $i\zeta(x) = iP/x + \pi\delta(x)$  is the Heitler function [13],[14]. represents k-summation in analogy with Cauchy principal value [15]. Introducing the operators (14) in equation (3) and eliminating the boson operators of EMF, it is obtain the following equation for operator  $O(t)$  in Born-Marcov approximation

$$\begin{aligned}
\frac{d}{dt}\langle O(t) \rangle &= \sum_k \sum_{l,j=1}^{N_r} \frac{(\mathbf{d}_a, \mathbf{g}_k)^2}{\hbar^2} \langle [R_j^+(t), O(t)] R_l^-(t) \rangle \\
&\times \exp[i(\mathbf{k}, \mathbf{r}_j - \mathbf{r}_l)] i\zeta^*(\omega_r - \omega_k) + \sum_k \sum_{l,j=1}^{N_s} \frac{(\mathbf{d}_a, \mathbf{g}_k)(\mathbf{d}_b, \mathbf{g}_k)}{\hbar^2} \\
&\times \langle [S_j^+(t), O(t)] S_l^-(t) \rangle \exp[i(\mathbf{k}, \mathbf{r}_j - \mathbf{r}_l)] i\zeta^*(\omega_s - \omega_k) \\
&+ \sum_{k,k'} \sum_{l,j=1}^N \frac{(\mathbf{n}_{eg}, \mathbf{e}_\lambda)^2 (\mathbf{n}_{ei}, \mathbf{e}_{\lambda'})^2 q^2(\omega_k, \omega_{k'})}{\hbar^2} \\
&\times \langle [D_j^+(t), O(t)] D_l^-(t) \rangle \exp[i(\mathbf{k} - \mathbf{k}', \mathbf{r}_j - \mathbf{r}_l)] i\zeta^*(\omega_a - \omega_k - \omega_{k'}) \\
&+ 2i \sum_{k,k'} \sum_{n=1}^N \sum_{l=1}^{N_s} \sum_{l=1}^{N_r} \frac{(\mathbf{d}_a, \mathbf{g}_{k'}) (\mathbf{d}_b, \mathbf{g}_k) (\mathbf{n}_{eg}, \mathbf{e}_\lambda) (\mathbf{n}_{ei}, \mathbf{e}_{\lambda'}) q(\omega_k, \omega_{k'})}{\hbar^3} \\
&\times \langle [D_n^+(t), O(t)] R_j^-(t) S_l^-(t) \rangle \\
&\times \exp[i(\mathbf{k}, \mathbf{r}_n - \mathbf{r}_l) + i(\mathbf{k}', \mathbf{r}_n - \mathbf{r}_l)] i\zeta^*(\omega_r - \omega_k) i\zeta^*(\omega_s - \omega_{k'}) \\
&- i \sum_{k,k'} \sum_{m=1}^N \sum_{l=1}^{N_s} \sum_{j=1}^{N_r} \frac{(\mathbf{d}_a, \mathbf{g}_{k_1}) (\mathbf{d}_b, \mathbf{g}_{k_2}) (\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1}) (\mathbf{n}_{ei}, \mathbf{e}_{\lambda_2}) q(\omega_1, \omega_2)}{\hbar^3} \\
&\times \langle [S_l^+(t) [R_j^+(t), O(t)] D_m^-(t)] i\zeta^*(\omega_r - \omega_k) i\zeta(\omega_s - \omega_{k'}) \rangle \\
&+ \langle R_l^+(t) [S_j^+(t), O(t)] D_m^-(t) \rangle i\zeta^*(\omega_s - \omega_k) i\zeta(\omega_r - \omega_{k'}) \\
&\times \exp[i(\mathbf{k}, \mathbf{r}_j) + i(\mathbf{k}', \mathbf{r}_l) - i(\mathbf{k} + \mathbf{k}', \mathbf{r}_m)] + H.C.(O^+ \rightarrow O). \tag{15}
\end{aligned}$$

In the right hand part of the equation (15) the third order terms contain the resonances between the single photon radiators  $A, B$  and two-photon radiator  $D$  described by the correlation functions  $\langle S_l^+(t) [R_j^+(t), O(t)] D_m^-(t) \rangle$ ,  $\langle [D_n^+(t), O(t)] R_j^-(t) S_l^-(t) \rangle$  and  $\langle R_l^+(t) [S_j^+(t), O(t)] D_m^-(t) \rangle$ . As it is observed from equation (15), these terms contain the product of the functions  $[P/(\omega_b - \omega_k)][P/(\omega_b - \omega_{k'})]$  which describe the principal value in the integration procedure on the variables  $k$  and  $k'$ . It is not difficult to observe that these integrals become divergent expressions. In order to avoid these divergence in Appendix1 it is proposed the integration procedure which takes in to account the retardation between the radiators in the representation of right hand side of equation (13). According with Appendix1 the right hand side of master equation (13) takes the following non-divergent form

$$\begin{aligned}
\frac{d}{dt}\langle O(t) \rangle &= \frac{1}{2\tau_r} \sum_{l,j=1}^{N_r} \chi_r(j, l) \langle [R_j^+(t), O(t)] R_l^-(t) \rangle + \frac{1}{2\tau_s} \sum_{l,j=1}^{N_s} \chi_s(j, l) \langle [S_j^+(t), O(t)] S_l^-(t) \rangle \\
&+ \frac{1}{2\tau_b} \sum_{l,j=1}^N \chi_d(j, l) \langle [D_j^+(t), O(t)] D_l^-(t) \rangle \\
&+ \frac{i}{2\tau_{bsr}} \sum_{m=1}^N \sum_{l=1}^{N_r} \sum_{l=0}^{N_s} U(j, l, m) \langle [D_m^+(t), O(t)] R_j^-(t) S_l^-(t) \rangle
\end{aligned}$$



$$\begin{aligned}
& - \frac{i}{4\tau_{sbr}} \sum_{m=1}^N \sum_{j=1}^{N_r} \sum_{l=0}^{N_s} V(j, l, m) [\langle S_l^+(t) [R_j^+(t), O(t)] D_m^-(t) \rangle + \langle R_j^+(t) [S_l^+(t), O(t)] D_m^-(t) \rangle] \\
& + H.C. (O^+ \rightarrow O).
\end{aligned} \tag{16}$$

Here the spontaneous emission  $\tau_i$  and exchange integral between radiators  $j$  and  $l$ ,  $\chi_i(j, l)$  are defined in expressions (28), (32), (34) and (37) of Appendix. This equation can be used for description of interaction between the dipole forbidden and dipole active systems of radiators.

### 3 Kinetic equations for correlation functions

In order to found the correlation process between dipole forbidden transitions of  $D$  subsystem and dipole active transitions in  $S$  and  $R$  subsystems of radiators let us found the equations for arbitrary atomic correlation functions of subsystems of radiators. According with the generalized equation (16) it is obtain the following chain of equation for atomic correlations

$$\begin{aligned}
\frac{d}{dt} \langle R_{zj}(t) \rangle &= -\frac{1}{2\tau_r} \sum_{l=1}^{N_a} [\chi_r(j, l) \langle R_j^+(t) R_l^-(t) \rangle + \chi_r^*(j, l) \langle R_l^+(t) R_j^-(t) \rangle] \\
&+ \frac{i}{4\tau_{sbr}} \sum_{m=1}^N \sum_{l=0}^{N_s} [V(j, l, m) \langle S_l^+(t) R_j^+(t) D_m^-(t) \rangle - V^*(j, l, m) \langle D_m^+(t) R_j^-(t) S_l^-(t) \rangle], \tag{17}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle S_{zl}(t) \rangle &= -\frac{1}{2\tau_s} \sum_{p=0}^{N_s} \chi_s(l, p) [\langle S_l^+(t) S_p^-(t) \rangle + \chi_s^*(l, p) \langle S_p^+(t) S_l^-(t) \rangle] \\
&+ \frac{i}{4\tau_{sbr}} \sum_{m=1}^N \sum_{l=0}^{N_s} [V(j, l, m) \langle S_l^+(t) R_j^+(t) D_m^-(t) \rangle - V^*(j, l, m) \langle D_m^+(t) R_j^-(t) S_l^-(t) \rangle], \tag{18}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle D_{zn}(t) \rangle &= -\frac{1}{2\tau_b} \sum_{l=1}^N [I_b(j, l) \langle D_n^+(t) D_l^-(t) \rangle + I_b^*(j, l) \langle D_l^+(t) D_n^-(t) \rangle] \\
&- \frac{i}{2\tau_{srb}} \sum_{j=1}^{N_r} \sum_{l=0}^{N_s} [U(j, l, n) \langle D_n^+(t) R_j^-(t) S_l^-(t) \rangle - U^*(j, l, n) \langle S_l^+(t) R_j^+(t) D_n^-(t) \rangle], \tag{19}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle R_j^+(t) R_l^-(t) \rangle &= \frac{1}{\tau_r} \sum_{n=1}^{N_r} [\chi_r(l, n) \langle R_j^+(t) R_{zl}(t) R_n^-(t) \rangle + \chi_r(n, j) \langle R_n^+(t) R_{zj}(t) R_l^-(t) \rangle] \\
&- \frac{1}{2\tau_{sbr}} \sum_{m=1}^N \sum_{k=0}^{N_s} [V(j, k, m) \langle S_k^+(t) R_j^+(t) R_{zl}(t) D_m^-(t) \rangle \\
&+ V^*(j, k, m) \langle D_m^+(t) R_{zl}(t) R_j^-(t) S_k^-(t) \rangle]; \tag{20}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \langle S_j^+(t) S_l^-(t) \rangle &= \frac{1}{\tau_r} \sum_{n=1}^{N_s} [\chi_r(l, n) \langle S_j^+(t) S_{zl}(t) S_n^-(t) \rangle + \chi_r(n, j) \langle S_n^+(t) S_{zj}(t) S_l^-(t) \rangle] \\
&- \frac{i}{2\tau_{sbr}} \sum_{m=1}^N \sum_{k=1}^{N_r} [V(k, l, m) \langle R_k^+(t) S_j^+(t) S_{zl}(t) D_m^-(t) \rangle \\
&- V^*(k, j, m) \langle D_m^+(t) S_{zj}(t) S_l^-(t) R_k^-(t) \rangle]; \tag{21}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\langle D_n^+(t)D_l^-(t) \rangle &= \frac{1}{\tau_b} \sum_{m=1}^N [I_b(l, m)\langle D_n^+(t)D_{zl}(t)D_m^-(t) \rangle + I_b^*(n, m)\langle D_m^+(t)D_{zn}(t)D_l^-(t) \rangle] \\
&+ \frac{i}{\tau_{bsr}} \sum_{j=1}^{N_r} \sum_{k=0}^{N_s} [U(j, k, m)\langle D_n^+(t)D_{zl}(t)R_j^-(t)S_k^-(t) \rangle \\
&+ U^*(j, k, m)\langle S_k^+(t)R_j^+(t)D_{zn}(t)D_l^+(t) \rangle].
\end{aligned} \tag{22}$$

$$\begin{aligned}
\frac{d}{dt}i\langle D_m^+(t)S_l^-(t)R_k^-(t) \rangle &= \frac{1}{\tau_{bsr}} \sum_{j=1}^{N_r} \sum_{n=1}^{N_s} [U^*(j, n, m)\langle S_n^+(t)R_j^+(t)D_{zm}(t)S_l^-(t)R_k^-(t) \rangle \\
&+ \frac{1}{2\tau_{sbr}} \sum_{n=1}^N \sum_{j=1}^{N_s} [V(j, l, n)\langle S_{zl}(t)R_j^+(t)R_k^-(t)D_m^+(t)D_n^-(t) \rangle \\
&+ V(k, j, n)\langle D_m^+(t)D_n^-(t)R_{zk}(t)S_j^+(t)S_l^-(t) \rangle] \\
&+ i \sum_{j=1} \left[ \frac{1}{\tau_s} \chi_r(j, l)\langle D_m^+(t)S_{zl}(t)S_j^-(t)R_k^-(t) \rangle \right. \\
&+ \frac{1}{\tau_r} \chi_r(j, k)\langle D_m^+(t)R_{zk}(t)S_l^-(t)R_j^-(t) \rangle \\
&\left. + \frac{1}{\tau_b} I_b^*(m, l)\langle D_j^+(t)D_{zm}(t)S_j^-(t)R_k^-(t) \rangle \right]
\end{aligned} \tag{23}$$

Let us consider the interaction three different atoms in interaction through vacuum EMF. Introducing the excited numbers for the atomic subsystems  $\langle N_\alpha \rangle = \langle J_{z\alpha}(t) \rangle + 0.5$  (here  $J \leftrightarrow S, R, D$   $\alpha = s, r, d$ ) and correlation function between the atoms  $\langle F \rangle = i[\langle D^+(t)S^-(t)R^-(t) \rangle - \langle S^+(t)R^+(t)D^-(t) \rangle]$  we can obtain the closed system of equations from the chain of equations (17-23). Indeed considering that the distance between the radiators is smaller than radiation wavelength:  $\Re \{U(j, k, m)\} = \Re \{V(j, k, m)\} = 1$ , we obtain the following closed system of equation

$$\begin{aligned}
\frac{d}{dt}\langle N_s(t) \rangle &= -\frac{\langle N_s \rangle}{\tau_s} - \frac{1}{2\tau_{sbr}}\langle F \rangle; \\
\frac{d}{dt}\langle N_r(t) \rangle &= -\frac{\langle N_r \rangle}{\tau_r} - \frac{1}{2\tau_{sbr}}\langle F \rangle; \\
\frac{d}{dt}\langle N_d(t) \rangle &= -\frac{\langle N_d \rangle}{\tau_d} - \frac{1}{\tau_{sbr}}\langle F \rangle; \\
\frac{d}{dt}\langle F(t) \rangle &= -\left[ \frac{1}{2\tau_s} + \frac{1}{2\tau_r} + \frac{1}{2\tau_d} \right] \langle F(t) \rangle \\
&+ \frac{1}{\tau_{bsr}} [6\langle N_s N_r N_d \rangle - 2\langle N_s N_r \rangle - \langle N_s N_d \rangle - \langle N_r N_d \rangle], \\
\frac{d}{dt}\langle N_s N_r N_d \rangle &= -\left[ \frac{1}{\tau_r} + \frac{1}{\tau_b} + \frac{1}{\tau_s} \right] \langle N_s N_r N_d \rangle, \\
\frac{d}{dt}\langle N_s N_r \rangle &= -\left[ \frac{1}{\tau_r} + \frac{1}{\tau_s} \right] \langle N_s N_r \rangle, \\
\frac{d}{dt}\langle N_s N_d \rangle &= -\left[ \frac{1}{\tau_r} + \frac{1}{\tau_b} \right] \langle N_s N_d \rangle, \\
\frac{d}{dt}\langle N_r N_d \rangle &= -\left[ \frac{1}{\tau_r} + \frac{1}{\tau_b} \right] \langle N_r N_d \rangle,
\end{aligned}$$

in which the new correlation functions between the atomic excitation is introduced  $\langle \hat{N}_s \hat{N}_r \hat{N}_d \rangle$ ,  $\langle \hat{N}_s \hat{N}_r \rangle$ ,  $\langle \hat{N}_s \hat{N}_d \rangle$ , and  $\langle \hat{N}_r \hat{N}_d \rangle$ . This system of equation is exactly solvable. The solution is

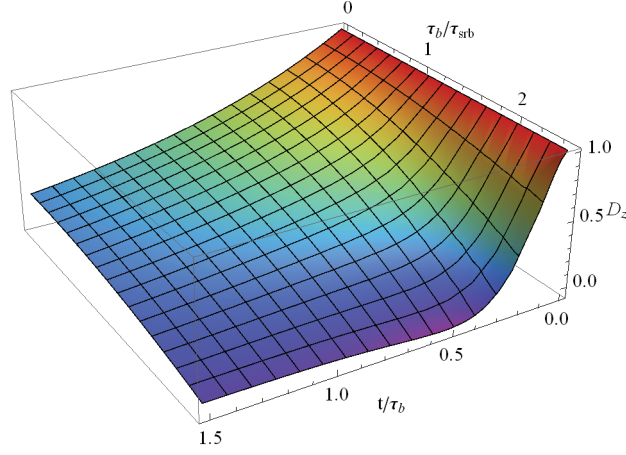


Figure 2: The decay law of inversion,  $D_z(t)$ , of dipole forbidden radiator stimulated by single photon decay processes of two radiators for following parameters of the system: relative decay rates of S- and R- atoms  $\tau_b/\tau_s = \tau_b/\tau_r = 5$ .

$$\begin{aligned}
\langle N_s N_r N_d \rangle &= \exp(-At); \quad \langle N_s N_r \rangle = \exp(-Bt), \langle N_s N_d \rangle = \exp(-Ct); \quad \langle N_d N_r \rangle = \exp(-Dt) \\
\langle N_i N_j \rangle &= \exp\left[-\left(\frac{1}{\tau_i} + \frac{1}{\tau_j}\right)t\right], \quad i, j \rightarrow s, r, b; \\
\langle F(t) \rangle &= \frac{\exp(-At/2)}{\tau_{bsr}} \left[ 6 \frac{1 - \exp[-At/2]}{A} - 4 \frac{1 - \exp[-(B - \tau_b^{-1})t/2]}{(B - \tau_b^{-1})} \right. \\
&\quad \left. - \frac{2(1 - \exp[-(C - \tau_r^{-1})t/2])}{(C - \tau_r^{-1})} - \frac{2(1 - \exp[-(C - \tau_s^{-1})t/2])}{(C - \tau_s^{-1})} \right]; \\
\langle N_i \rangle &= \exp[-t/\tau_i] - \frac{1}{\tau_{bsr}} \int_0^t \exp[-(t-t')/\tau_i] \langle F(t') \rangle, \quad i \equiv s, r, d,
\end{aligned}$$

where the collective rates are defined  $A = 1/\tau_r + 1/\tau_b + 1/\tau_s$ ;  $B = 1/\tau_r + 1/\tau_s$ ;  $C = 1/\tau_b + 1/\tau_s$ ;  $D = 1/\tau_b + 1/\tau_s$ . The solution of this system of equation is plotted in figure 2. It is observed the influence of single photon transition on the two quanta transitions. This influence drastically depends on the cooperative rate  $1/\tau_{bsr}$ .

As follows from figure 3 the cooperative exchanges between the radiator accelerate the two-photon decay processes so that the the derivation  $-\frac{dD_z}{dt}$  achieved the maximal value decreasing after that till zero value.

Neglecting the quantum fluctuations of inversion operators  $\langle R_{zj} \rangle$ ,  $\langle S_{zl} \rangle$ ,  $\langle D_{zn} \rangle$ , dipole-dipole correlations between the same radiators  $\langle R_l^+(t) R_j^-(t) \rangle$ ,  $\langle S_l^+(t) S_p^-(t) \rangle$  and  $\langle D_n^+(t) D_l^-(t) \rangle$  and between subsystems  $\langle S_l^+(t) R_j^+(t) D_m^-(t) \rangle$ ,  $\langle D_m^+(t) R_j^-(t) S_l^-(t) \rangle$  we can de-correlated the chain of equations (17-23) in order to obtain the closed system of equations

$$\begin{aligned}
\text{for } J \leftrightarrow S, R, D; \quad \langle J_j^+(t) J_{zl}(t) J_m^-(t) \rangle &= \langle J_{zl}(t) \rangle \langle J_j^+(t) J_m^-(t) \rangle \quad j \neq l \neq m; \\
\langle S_k^+(t) R_j^+(t) R_{zl}(t) D_m^-(t) \rangle &= \langle R_{zl}(t) \rangle \langle S_k^+(t) R_j^+(t) D_m^-(t) \rangle_{l \neq j} - \delta_{l,j} \langle S_k^+(t) R_j^+(t) D_m^-(t) \rangle; \\
\langle D_n^+(t) D_{zl}(t) R_j^-(t) S_k^-(t) \rangle &= \langle D_{zl}(t) \rangle \langle D_n^+(t) R_j^-(t) S_k^-(t) \rangle_{l \neq n} - \delta_{l,n} \langle D_n^+(t) R_j^-(t) S_k^-(t) \rangle; \\
\langle S_l^+(t) R_j^+(t) D_{zn}(t) R_p^-(t) S_k^-(t) \rangle &= \langle D_{zn}(t) \rangle \langle R_j^+(t) R_p^-(t) \rangle \langle S_l^+(t) S_k^-(t) \rangle_{l \neq k; j \neq p}
\end{aligned}$$

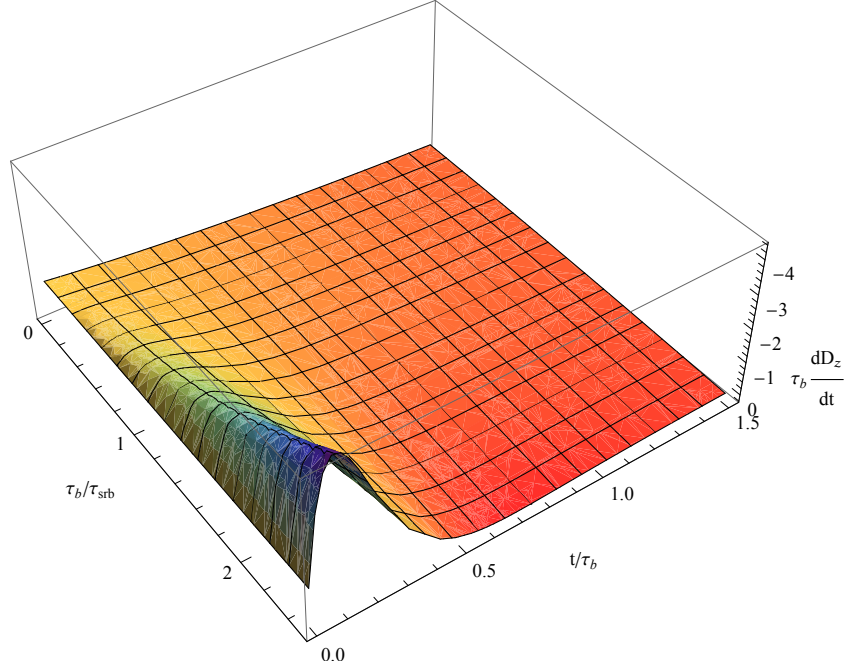


Figure 3: The two-photon decay rate  $dD_z(t)/dt$  stimulated by single photon processes for same parameter of the system as in figure 2.

$$\begin{aligned}
& +\delta_{j,p}\delta_{l,k}\langle D_{zn}(t)(R_{zj}(t)+0.5)(S_{zl}(t)+0.5)\rangle; \\
\langle D_m^+(t)D_n^-(t)S_{zl}(t)R_j^+(t)R_k^-(t)\rangle &= \langle S_{zl}(t)\rangle\langle D_m^+(t)D_n^-(t)\rangle\langle R_j^+(t)R_k^-(t)\rangle_{m\neq n;j\neq k} \\
& +\delta_{m,n}\delta_{j,k}\langle (D_{zn}(t)+0.5)S_{zl}(t)(R_{zj}(t)+0.5)\rangle; \\
\langle D_m^+(t)D_n^-(t)R_{zj}(t)S_l^+(t)S_k^-(t)\rangle &= \langle R_{zj}(t)\rangle\langle D_m^+(t)D_n^-(t)\rangle\langle S_l^+(t)S_k^-(t)\rangle_{m\neq n;l\neq k} \\
& \delta_{m,n}\delta_{l,k}\langle (D_{zn}(t)+0.5)R_{zj}(t)(S_{zl}(t)+0.5)\rangle. \tag{24}
\end{aligned}$$

Taking in to account the de-correlation (24), we obtain the following closed system of equations

$$\begin{aligned}
\frac{d}{dt}R_z(t) &= -\frac{1}{\tau_r}\{N_r(N_r+2)/4 - R_z^2 + R_z\} - \frac{1}{2\tau_{sbr}}F, \\
\frac{d}{dt}S_z(t) &= -\frac{1}{\tau_r}\{N_s(N_s+2)/4 - S_z^2 + S_z\} - \frac{1}{2\tau_{sbr}}F, \\
\frac{d}{dt}D_z(t) &= -\frac{1}{\tau_r}\{N(N+2)/4 - D_z^2 + D_z\} - \frac{1}{\tau_{sbr}}F; \\
\frac{d}{dt}F &= \left[\frac{1}{\tau_s}[S_z(t)-1] + \frac{1}{\tau_r}[R_z(t)-1] + \frac{1}{\tau_d}[D_z(t)-1]\right]F \\
&+ \frac{1}{\tau_{bsr}}[2D_z\{N_s^2/4 - S_z^2\}\{N_r^2/4 - R_z^2\} \\
&+ R_z\{N_s^2/4 - S_z^2\}\{N^2/4 - D_z^2\} + S_z\{N_r^2/4 - R_z^2\}\{N^2/4 - D_z^2\} \\
&+ (4N_sN_rN \exp[-A*t] - N_s * N_r * \exp[-Bt] - 0, 5N_sN \exp[-Ct] \\
&- 0.5N_rN \exp[-Dt]].
\end{aligned}$$

From this system of equations follows the oscillatory behavior of the decay rate of the inversion  $D_z$ . Taking in to account the following relative expressions of the decay rates we obtain the numerical simulation of the inversion  $D_z$  and its derivative (see figure 4 and figure 5). As follows from this

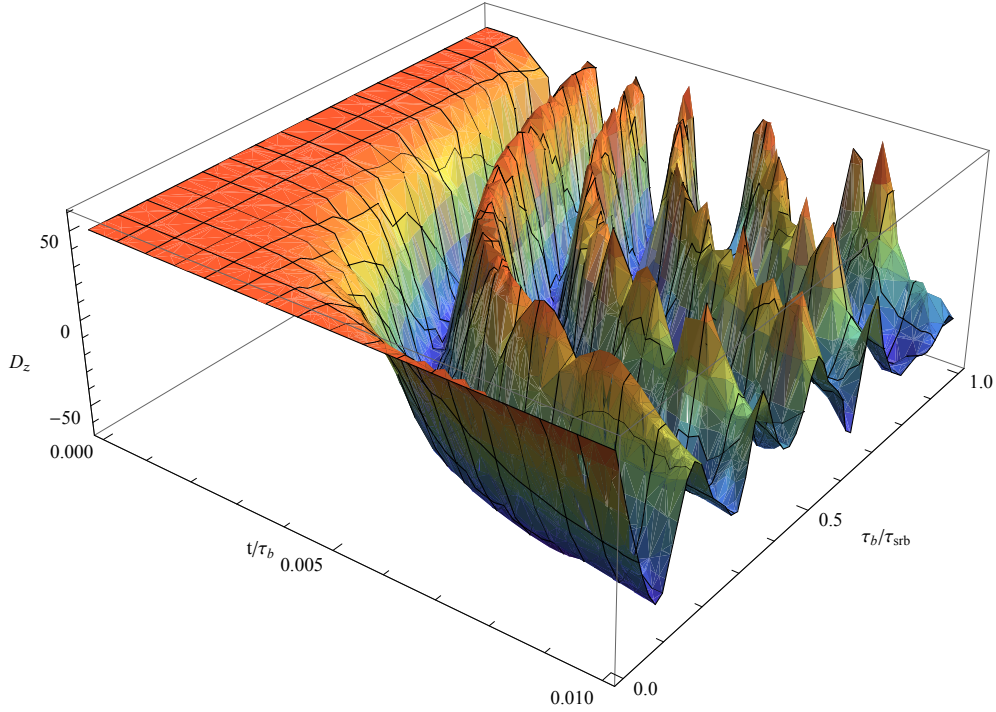


Figure 4: The time dependence of inversion  $D_z(t)$  for dipole forbidden radiator subsystem for following values of the parameter of the system: number of atoms in the subsystems  $S, R$  and  $D$  are,  $N_s = N_r = N = 50$  respectively; the relative decay times of the subsystems are  $\tau_b/\tau_s = \tau_b/\tau_r = 6$ ; the coupling parameter of these three system is changed between 0 and 1. The oscillatory behavior of decay rate is observed.

system of equation the increasing of decay rate of two-photon spontaneous emission is possible under the influence of single photon cooperative emission of two atomic subsystems. In figures 4 is plotted the time dependence of the inversion  $\langle D_z(t) \rangle$  of dipole forbidden radiators as function of the relative coupled parameter between the radiators  $\tau_b/\tau_{srb}$ .

The same dependence is represented in figure 5 for the intensity of two-photon emission proportional to  $-d \langle D_z(t) \rangle / dt$ . As follows from these plots it is observed the mutual influences between single and two-photon super-radiance processes of three particle interaction. This effect plays an important role in the collective decay process of the systems of radiators with the dimension smaller than wavelength.

## 4 Conclusion

In this paper the effective interaction between three radiator subsystems in two-photon resonance is found using the method of elimination of operators of vacuum field. The new cooperative interaction between dipole-forbidden atomic subsystem and two-dipole active subsystems of radiators was proposed. The master equation 16, which describes the energy dissipation from the system due to mutual interaction between the radiators through the vacuum of electromagnetic field, was obtained. Using the chain of equation 17-23, which describes the cooperative interaction between three radiator subsystems, it is

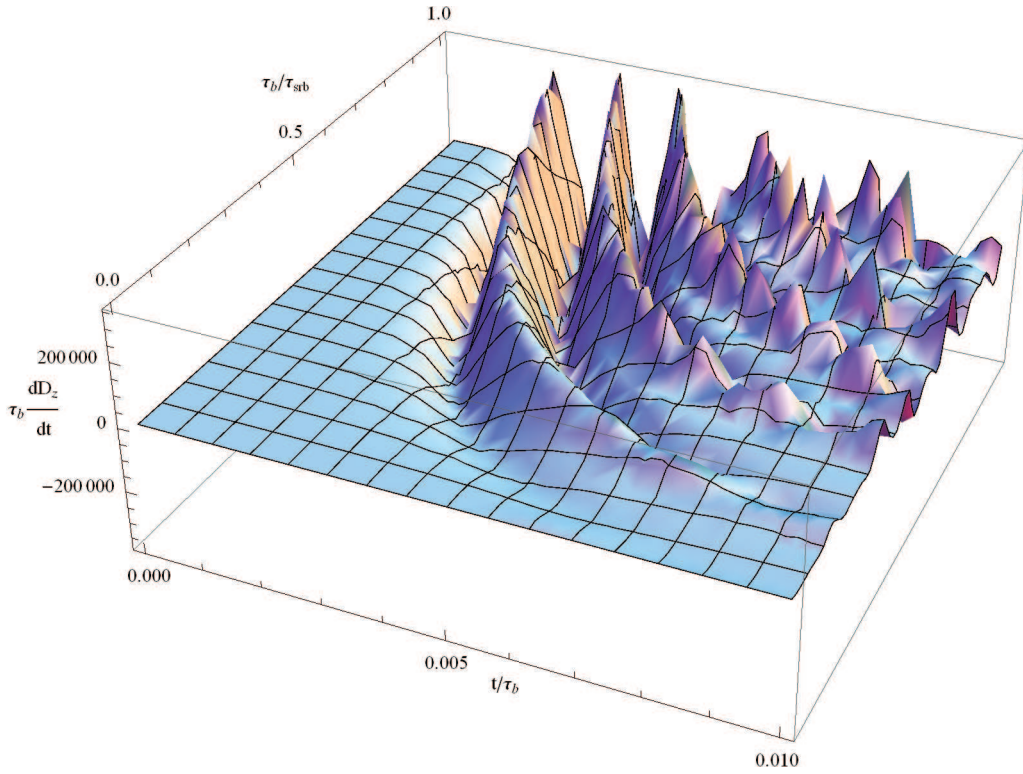


Figure 5: The cooperative decay rate of inversion  $\frac{dD_z(t)}{dt}$  of dipole forbidden radiator subsystem for same values of the parameters of the system as in figure 4. The increasing of decay rate of bi-photons is observed.

obtained the closed system of equations for three radiators. Neglecting the quantum fluctuation of the inversion, the de-correlation method of the this chain of equation is proposed 24 in order to describe numerically the behavior of mutual influences of single and two-photon super-radiance processes. As a consequence of effective interaction between three radiators through two-photon resonance processes of inverted systems increase substantially in process cooperative decay of the system. The three particle exchange integral has been established and the influence of this effect on the behavior cooperative decay of the atomic subsystems was estimated (see figure 4 and figure 5). Similar experimental situation can be realized in excited atomic (for example transitions in Cs atoms [17]) or nuclei (for example  $^{193m}\text{Ir}$ ,  $^{195m}\text{Pt}$  and  $^{103m}\text{Rh}$  nuclei[16]) subsystems in resonance interaction through vacuum field.

## 5 Appendix: Exchange integrals

In order to estimate all exchange integrals in equation (13) let us firstly find the well known exchange integral between two radiators in single photon interaction with vacuum of electromagnetic field. In the first terms of equation (13) the retardation can be found integration firstly on the  $k$  vector respectively

$$V_{jl}^i = \frac{d_\alpha^2}{(2\pi)^2 \hbar c^3} \sum_{l,j=1}^{N_a} \int_0^\infty \omega_k^3 d\omega_k \int d\Omega_k \int_0^t d\tau \exp[i(\omega_i - \omega_k)\tau] \\ \times (1 - (\mathbf{e}_k, \mathbf{n}_d)) \langle [\tilde{J}_j^+(t), O(t)] \tilde{J}_l^-(t - \tau) \rangle \exp[i\omega_k r_{jl} \cos \theta],$$

$$i \equiv a, b. \quad (25)$$

Here the frequency  $\omega_i$  corresponds to  $A$  and  $B$  atomic systems;  $i = r, s$ ; for  $i = r$  operators  $\tilde{J}_j^+$ ,  $\tilde{J}_l^-$  corresponds to  $\tilde{R}_j^+$ ,  $\tilde{R}_l^-$  and for  $i = s$  these operators corresponds to  $\tilde{S}_j^+$ ,  $\tilde{S}_l^-$ . Passing to new variable  $v = \omega_k - \omega_i$  and considering that the smooth function  $\omega_k$  under integral can be approximation with  $\omega_i^3$ , we obtain the following approximate expression of thirist order exchange integrals

$$\begin{aligned} \omega_i^3 \int_{-\omega_k}^{\infty} dv \exp[iv(\tau - r_{jl} \cos \theta / c)] &\approx \omega_i^3 \int_{-\infty}^{\infty} dv \exp[iv(\tau - r_{jl} \cos \theta / c)] \\ &= 2\pi\omega_i^3 \delta(\tau - r_{jl} \cos \theta / c). \end{aligned} \quad (26)$$

$\omega_i$  is the emission frequency relatively the dipole active transitions of the  $R$  and  $S$  atomic subsystems. In this approximation I obtain the following integral on angle  $\theta$  and retardation  $\tau$

$$\begin{aligned} V_{jl}^i &= \frac{\omega_i^3 d_i^2}{2\hbar c^3} \int_0^\pi d\theta \int_0^t d\tau \sin \theta \delta(\tau - r_{jl} \cos \theta / c) D_i(\theta) \langle [\tilde{J}_j^+(t), O(t)] \tilde{J}_l^-(t) \rangle \\ &= \frac{\omega_i^3 d_i^2}{2\hbar c^3} \int_0^\pi d\theta \sin \theta \Theta(\cos \theta) D_{jl} \left[ \frac{\partial}{\partial \omega_i} \right] \exp[i\omega_i r_{jl} \cos \theta] \langle [\tilde{J}_j^+(t), O(t)] \tilde{J}_l^-(t) \rangle \\ &= \frac{1}{2\tau_i} \chi(j, l) \langle [\tilde{J}_j^+(t), O(t)] \tilde{J}_l^-(t) \rangle. \end{aligned} \quad (27)$$

Here  $\tau_i$  and  $\chi(j, l)$  are the spontaneous emission time and exchange integral between the single photon radiators respectively [13]

$$\tau_i = \frac{3\hbar c^3}{4d_i \omega_i^3}, \quad \chi(j, l) = D_{jl} \left[ \frac{\partial}{\partial \omega_i} \right] \frac{3 \exp[i\omega_i r_{jl}/c] - 1}{i\omega_i r_{jl}/c}, \quad (28)$$

the expressions in equation (27) and (28) are defined below

$$\begin{aligned} D_i(\theta) &= 1 + \cos^2 \xi_i - \cos^2 \theta (3 \cos^2 \xi_i - 1), \\ D_{jl} \left[ \frac{\partial}{\partial \omega_i} \right] &= [1 + \cos^2(\xi_i) + (3 \cos^2 \xi_i - 1) \frac{c^2}{r_{jl}^2} \frac{\partial^2}{\partial \omega_i^2}]^2 \end{aligned} \quad (29)$$

where  $\cos \xi_i$  is the scalar product between the unitary vectors along the direction of dipole momentum of the  $j$  (or  $l$ ) atom  $\mathbf{n}_{d_i} = \mathbf{d}_i/d_i$  and the direction of the distance between the  $j$  and  $l$  atoms  $\mathbf{n}_{jl} = \mathbf{r}_{jl}/r_{jl}$ .

**2.** The two-photon exchange integral between dipole forbidden transition of the radiators of  $D$  subsystem is described by third term in the right hand site of equation (13)

$$\begin{aligned} V_{jl}^b &= \frac{V^2}{(2\pi)^6} \int_0^{2\pi} d\varphi_1 \int_0^\pi d\theta_1 \sin \theta_1 \int_0^\infty k_1^2 dk_1 \int_0^{2\pi} d\varphi_2 \int_0^\pi d\theta_2 \sin \theta_2 \int_0^\infty k_2^2 dk_2 \frac{(\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1})^2 (\mathbf{n}_{ei}, \mathbf{e}_{\lambda_2})^2 q^2(\omega_1, \omega_2)}{\hbar^2} \\ &\times \int_0^t d\tau \exp[-i(2\omega_0 - \omega_1 - \omega_2)\tau + i(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{r}_j - \mathbf{r}_l)] \left\langle [\tilde{D}_j(t), O(t)] \tilde{D}_l(t - \tau) \right\rangle. \end{aligned} \quad (30)$$

The exchange integral in Born-Marcov approximation for two-photon emission was obtained in paper [9]. Here we will estimate the exchange integral of expression (30) integration firstly on the wave vectors  $\mathbf{k}_1$  and  $\mathbf{k}_2$ . Indeed considering that the amplitude  $q^2(\omega_1, \omega_2)$  is the smooth function of the variables  $k_1$  and  $k_2$  in comparison with rapid oscillation functions  $\exp[i(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{r}_j - \mathbf{r}_l)]$  and  $\exp[-i(2\omega_0 - \omega_{k_1} - \omega_{k_2})\tau]$ , we can approximate the amplitude  $q^2(\omega_1, \omega_2)$  according with the resonance frequencies of the atomic

system  $q^2(\omega_1, \omega_2) \approx q^2(\omega_0, \omega_0)$ . It is not difficult to observe that the maximal value of the smooth function under the integral is obtained for frequencies  $\omega_{k_2} = \omega_{k_1} \approx \omega_0$ . In order to integrate this expression on the variables  $k_1$  and  $k_2$  we change the variables  $x_i = \omega_{k_i} - \omega_i$  as in expression (26), and consider that  $\tilde{D}_j^+(t) = D_j^+(t) \exp[-2i\omega_0 t]$  is smooth operator. In these approximations it is obtain the following expression for  $V_{jl}^b$

$$\begin{aligned} V_{jl}^b &= \frac{V^2 q^2(\omega_0, \omega_0)}{(2)^4 \pi^2} \int_0^t d\tau \int_0^\pi d\theta_1 \sin \theta_1 \int_0^\pi d\theta_2 \sin \theta_2 D_0(\theta_1) D_0(\theta_2) \exp[i\omega_0 r_{jl}(\cos \theta_1 + \cos \theta_2)/c] \\ &\quad \delta(\tau - r_{jl} \cos \theta_1/c) \delta(\tau - r_{jl} \cos \theta_2/c) \langle [\tilde{D}_j(t), O(t)] \tilde{D}_l(t - \tau) \rangle \\ &\approx \frac{V^2 q^2(\omega_0, \omega_0)}{(2)^4 \pi^2} D_{jl}^2 \left( \frac{\partial}{\partial \omega_0} \right) \frac{\exp[2i\omega_0 r_{jl}] - 1}{2i\omega_0 r_{jl}} \langle [\tilde{D}_j(t), O(t)] \tilde{D}_l(t - \tau) \rangle, \end{aligned} \quad (31)$$

where expressions  $D_0(\theta_1)$  and  $D_{jl}(\partial/(\partial \omega_0))$  are defined by the expressions (29). Integrating the right hand site of the equation (31) on the solid angle and retardation, we obtain the following approximative expression

$$V_{jl}^b = \frac{1}{2\tau_b} \chi_b(j, l) \langle [\tilde{D}_j^+(t), O(t)] \tilde{D}_l^-(t) \rangle,$$

where

$$\begin{aligned} \frac{1}{2\tau_b} &= \frac{2^2 \omega_0^7 d_{23}^2 d_{31}^2}{3^2 2^2 \pi \hbar^2 c^6} \{3/2\} \left\{ \frac{1}{\omega_{32} + \omega_0} + \frac{1}{\omega_{31} - \omega_0} \right\}^2, \\ \chi_b(j, l) &= \frac{3^2}{4} \frac{\pi c}{4\omega_0 r_{jl}} D^2 \left[ \frac{\partial}{\partial \omega_0} \right] \frac{\exp[2i\omega_0 r_{jl}/c] - 1}{i\omega_0 r_{jl}/c} \end{aligned} \quad (32)$$

This exchange integral diverges, when the distance between the radiators  $r_{jl}$  is less than the wavelength  $\lambda_0 = 2\pi c/\omega_0$ . In order to take in to account the value of the exchange integral for the small parameter,  $r_{jl}/\lambda_0 \ll 1$ , let us integrate the expression (30) taking in to account the method proposed in papers [9] and [12]. In this case we obtain the following expression for  $ReV_{jl}^b$

$$\begin{aligned} F(j, l) &= ReV_{jl}^b = \frac{d_{31}^2 d_{32}^2}{4\pi \hbar^2 c^6} \int_0^{2\omega_0} d\omega_k \omega_k^3 (\omega_{21} - \omega_k)^3 \\ &\quad \times \chi_{jl}(\omega_k) \chi_{jl}(\omega_{21} - \omega_k) \left\{ \frac{1}{\omega_{31} - \omega_{k_1}} + \frac{1}{\omega_{32} + \omega_{k_1}} \right\}^2, \end{aligned}$$

where

$$\chi_{jl}(\omega) = (1 - \cos^2 \xi) \frac{\sin \frac{\omega r_{jl}}{c}}{\frac{\omega r_{jl}}{c}} + (1 - 3 \cos^2 \xi) \left\{ \frac{\cos \frac{\omega r_{jl}}{c}}{(\frac{\omega r_{jl}}{c})^2} - \frac{\sin \frac{\omega r_{jl}}{c}}{(\frac{\omega r_{jl}}{c})^3} \right\}.$$

**3.** In the right hand part of this equation (13) the third order terms contains the resonances between dipole active radiators  $A, B$  and dipole forbidden radiators  $D$ , described by the correlation functions  $\langle S_l^+(t) [R_j^+(t), O(t)] D_m^-(t) \rangle$ ,  $\langle [D_n^+(t), O(t)] R_j^-(t) S_l^-(t) \rangle$  and  $\langle R_l^+(t) [S_j^+(t), O(t)] D_m^-(t) \rangle$ . Let us introduced the similar approximation in the chronological interaction between the atomic subsystems  $A, B$ , and  $D$ . The exchange integral between three atoms is represented in the similar form as in the expression (30)

$$V_{jln;as-d}^c = i \frac{V}{(2\pi)^3} \frac{V}{(2\pi)^3} \frac{2\pi \hbar d_s d_r}{V c^6 \hbar^3} \frac{(2\pi)^3}{V} (2\pi)^2 \left( \frac{1}{2} \right)^2 \int_0^\infty \omega_1^2 d\omega_1 \int_0^\infty \omega_2^2 d\omega_2 \sqrt{\omega_1 \omega_2} \chi(\omega_1, \omega_2)$$



$$\begin{aligned}
& \times \int_{-1}^1 dx_1 \int_{-1}^1 dx_2 \int_0^t d\tau_1 \int_0^t d\tau_2 \exp[-i(\omega_1 - \omega_r)\tau_1 - i(\omega_2 - \omega_s)\tau_2] \\
& \times D_{nl} \left[ \frac{\partial}{\partial \omega_s} \right] D_{nj} \left[ \frac{\partial}{\partial \omega_r} \right] \exp[i\omega_1 r_{nj} x_1 / c + i\omega_2 r_{nl} x_2 / c] \\
& \langle [\tilde{D}_n^+(t), O(t)] \tilde{R}_j^-(t - \tau_1) \tilde{S}_l^-(t - \tau_2) \rangle;
\end{aligned} \tag{33}$$

After the substitution of variables  $\omega_1 - \omega_r = \tilde{\omega}_1$  and  $\omega_2 - \omega_s = \tilde{\omega}_2$ , we can approximate the smooth amplitude  $\omega_1^2 \omega_2^2 \sqrt{\omega_1 \omega_2} \chi(\omega_1, \omega_2)$  with expression  $(\omega_r)^2 \omega_s^2 \sqrt{\omega_r \omega_s} \chi(\omega_r, \omega_s)$ . Integrals on the new variable  $\tilde{\omega}_1$  and  $\tilde{\omega}_2$  give the following aspect of expression (33)

$$\begin{aligned}
V_{jln;as-d}^c &= 2\pi i \left( \frac{1}{2} \right)^2 \frac{d_s d_r}{c^6 \hbar^2} \omega_s^2 (\omega_r)^2 \sqrt{\omega_r \omega_s} \chi(\omega_s, \omega_r) \int_0^1 dx_1 \int_0^1 dx_2 \\
&\times D_{nl} \left[ \frac{\partial}{\partial \omega_s} \right] D_{nj} \left[ \frac{\partial}{\partial \omega_r} \right] \exp[i\omega_r r_{nj} x_1 / c + i\omega_s r_{nl} x_2 / c] \\
&\times \langle [\tilde{D}_n^+(t), O(t)] \tilde{R}_j^-(t - r_{nj} x_1 / c) \tilde{S}_l^-(t - r_{nl} x_2 / c) \rangle
\end{aligned}$$

from which follows that  $x_1$  and  $x_2 > 0$ . In the Born approximation the expression for  $V_{jln;as-d}^c$

$$V_{rs-d}(m, j, l) = \frac{i}{4\tau_{bsr}} U(j, l, m) \langle [\tilde{D}_n^+(t), O(t)] \tilde{R}_j^-(t) \tilde{S}_l^-(t) \rangle$$

where

$$\begin{aligned}
\frac{1}{\tau_{bsr}} &= \left( \frac{2}{3} \right)^2 \frac{d_s d_r d_{23} d_{31} \omega_s^3 (\omega_r)^3}{4\pi c^6 \hbar^2} \left\{ \frac{1}{\omega_{32} + \omega_s} + \frac{1}{\omega_{31} - \omega_r} \right\}, \\
U(j, l, m) &= - \left( \frac{3}{2} \right)^2 D_{nl} \left[ \frac{\partial}{\partial \omega_s} \right] D_{nj} \left[ \frac{\partial}{\partial \omega_r} \right] \frac{c^2 [\exp i\omega_r r_{nj} / c] - 1 [\exp i\omega_s r_{nl} / c] - 1}{\omega_r \omega_s r_{nj} r_{nl}}.
\end{aligned} \tag{34}$$

**4.** Let now found the retardation in the last correlation function term of equation (13). Taking in to account the retardation in the rapid oscillation part of atomic operators  $J^\pm(t - \tau) = \tilde{J}^\pm(t - \tau) \exp[\pm i\omega(t - \tau)]$  where  $J^\pm$ ,  $\omega$  and  $\tau$  are the atomic operators, transition frequencies and delay time for atomic subsystems  $A$ ,  $B$  and  $D$  respectively,

$$\begin{aligned}
V_{ab-d}(t) &= i \sum_{k_1 k_2} \sum_{m=1}^N \sum_{l=1}^{N_a} \sum_{j=0}^{N_b} \frac{(\mathbf{d}_r, \mathbf{g}_{k_1})(\mathbf{d}_s, \mathbf{g}_{k_2})(\mathbf{n}_{eg}, \mathbf{e}_{\lambda_1})(\mathbf{n}_{ei}, \mathbf{e}_{\lambda_2}) q(\omega_{k_1}, \omega_{k_2})}{\hbar^3} \\
&\times \int_0^t d\tau_1 \exp[i(2\omega_0 - \omega_{k_1} - \omega_{k_2})\tau_1] \int_0^t d\tau_2 \exp[-i(\omega_s - \omega_{k_1})\tau_2] \\
&\times \exp[-i(\mathbf{k}_1, \mathbf{r}_j - \mathbf{r}_m) + i(\mathbf{k}_2, \mathbf{r}_l - \mathbf{r}_m)] \langle \tilde{S}_l^+(t - \tau_2) [\tilde{R}_j^+(t), O(t)] \tilde{D}_m^-(t - \tau_1) \rangle.
\end{aligned} \tag{35}$$

Passing from the summation to integration in expression (35) we obtain following expression for correlation between the  $j$ ,  $l$  and  $m$  atoms

$$\begin{aligned}
V_{rs-d}(m, j, l) &= i \frac{V}{(2\pi)^3} \frac{V}{(2\pi)^3} \frac{2\pi \hbar d_s d_r}{V c^6 \hbar^3} \frac{(2\pi)^3}{V} (2\pi)^2 \left( \frac{1}{2} \right)^2 \int_0^\infty \omega_1^2 d\omega_1 \int_0^\infty \omega_2^2 d\omega_2 \\
&\times \sqrt{\omega_1 \omega_2} \chi(\omega_1, \omega_2) \int_{-1}^1 dx_1 \int_{-1}^1 dx_2 \int_0^t d\tau_1 \exp[i(2\omega_0 - \omega_1 - \omega_2)\tau_1]
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^t d\tau_2 \exp[-i(\omega_s - \omega_2)\tau_2] \\
& \times D_{jl} \left[ \frac{\partial}{\partial \omega_2} \right] D_{jm} \left[ \frac{\partial}{\partial \omega_1} \right] \exp[-i\omega_2 r_{ml} x_2 / c - i\omega_1 r_{jm} x_1 / c] \\
& \times \langle \tilde{S}_l^+(t - \tau_2) [\tilde{R}_j^+(t), O(t)] \tilde{D}_m^-(t - \tau_1) \rangle.
\end{aligned} \tag{36}$$

Introducing the new variables  $u_1 = \omega_1 - \omega_r$ ;  $u_2 = \omega_2 - \omega_s$  in (36), and approximating the smooth amplitude  $\omega_1^2 \omega_2^2 \sqrt{\omega_1 \omega_2} \chi(\omega_1, \omega_2)$  with expression  $\omega_s^2 \omega_r^2 \sqrt{\omega_s \omega_r} \chi(\omega_s, \omega_r)$  we get using the delta functions (26)

$$\begin{aligned}
V_{jlm;as-d}(j, l) &= 2\pi i \left( \frac{1}{2} \right)^2 \frac{d_s d_r}{c^6 \hbar^2} \omega_s^2 \omega_r^2 \sqrt{\omega_s \omega_r} \chi(\omega_s, \omega_r) \int_{-1}^1 dx_1 \int_{-1}^1 dx_2 \int_0^t d\tau_1 \int_0^t d\tau_2 \\
&\times \delta(\tau_2 - \tau_1 - r_{jl} x_2 / c) \delta(\tau_1 - r_{jm} x_1 / c) \\
&\times D_{jl} \left[ \frac{\partial}{\partial \omega_s} \right] D_{jm} \left[ \frac{\partial}{\partial \omega_r} \right] \exp[i\omega_s r_{ml} x_2 / c] \exp[-i\omega_r r_{jm} x_1 / c] \\
&\times \langle \tilde{S}_l^+(t - \tau_2) [\tilde{R}_j^+(t), O(t)] \tilde{D}_m^-(t - \tau_1) \rangle,
\end{aligned}$$

the value of which can be estimated observing from the arguments of  $\delta$ -functions that  $x_1 > 0$  and  $r_{jm} x_1 + r_{jl} x_2 > 0$ . In this case, neglecting the retardation  $\tau_1$  and  $\tau_2$  in the smooth correlation function we obtain

$$V_{jlm;as-d}(j, l) = \frac{i}{4\tau_{sbr}} V(j, l, m) \langle \tilde{S}_l^+(t) [\tilde{R}_j^+(t), O(t)] \tilde{D}_m^-(t) \rangle,$$

where cooperative rate is

$$\frac{1}{\tau_{sbr}} = \left( \frac{2}{3} \right)^2 \frac{d_s d_r d_{23} d_{31} \omega_s^3 \omega_r^3}{4\pi c^6 \hbar^2} \left\{ \frac{1}{\omega_{32} + \omega_s} + \frac{1}{\omega_{31} - \omega_r} \right\}.$$

the integral  $V(j, l, m)$  on the direction of the emitted photons is

$$\begin{aligned}
V(j, l, m) &= \left( \frac{3}{2} \right)^2 D_{ml} \left[ \frac{\partial}{\partial \omega_s} \right] D_{jm} \left[ \frac{\partial}{\partial \omega_r} \right] \\
&\times \left[ \frac{c^2 \{ 2 \exp[i(\omega_s r_{ml} - \omega_r r_{jm})/c] - 2 \exp[2i\omega_s r_{ml}] - \exp[-2i\omega_r r_{jm}] + 1 \}}{2\omega_s \omega_r r_{lm} r_{ml}} \theta(r_{ml} - r_{mj}) \right. \\
&+ \left. \frac{(c^2 (\omega_s + \omega_r) [\exp(-i\omega_r r_{mj}/c) - 1] \exp(i\omega_s r_{ml}/c) - \exp[-i\omega_s r_{ml}/c - i\omega_r r_{mj}/c])}{(\omega_s + \omega_r) \omega_s \omega_r r_{mj} r_{ml}} \right. \\
&+ \left. \frac{c^2 [\omega_s \exp[-i(\omega_s + \omega_r) r_{ml}/c] + \omega_r]}{(\omega_s + \omega_r) \omega_r \omega_s r_{jm} r_{ml}} \right] \theta(r_{mj} - r_{jl}).
\end{aligned} \tag{37}$$

If we will consider that  $x_1 > 0$ , and  $x_2 > 0$  the expression for (37) takes more simple form

$$\begin{aligned}
V(j, l, m) &\approx \left( \frac{3}{2} \right)^2 D_{ml} \left[ \frac{\partial}{\partial \omega_s} \right] D_{jm} \left[ \frac{\partial}{\partial \omega_r} \right] \\
&\frac{c^2 \{ \exp[i\omega_s r_{ml}/c] - 1 \} [\exp[-i\omega_r r_{jm}/c] - 1]}{\omega_s \omega_r r_{lm} r_{ml}}
\end{aligned}$$

The expression for exchange integrals described in points 1. – 4. are used in the master equation (16)

## References

- [1] B. Nikolaus, D. Zhang, and P. Toschek, Phys. Rev. Lett. **47**, 171(1981).

- [2] D.J. Gauthier, Q. Wu, S.E. Morin, and T.W. Mossberg, Phys.Rev. Lett. 68, 464 (1992);
- [3] M.Bruno, J. M. Raimond, S. Haroche, Phys. Rev. A **35**, 3888 (1987); M. Bruno, J. M. Raimond, P. Goy, Phys. Rev. Lett. **59**,1898 (1987).
- [4] G.Breit and E.Teller, Astrophys. J . **91**, 215 (1940);
- [5] M. Goeppert-Mayer, Ann Phys. (Leipzig) **9**, 273 (1931).
- [6] L. Spitzer and J.L.Greenstain, Astrophys. J . **144**, 215 (1940).
- [7] J. Shapiro and G. Breit, Phys Rev. **113**, 179 (1959).
- [8] S.G.Alexander and P. M.Eszaros, Astrophysics J. **327**, 554 (1991); 565 (1991).
- [9] N.A.Enaki, Zh. Eksp. Teor.Fiz.**94**, 135 (1988); N.A.Enaki, Zh. Eksp. Teor.Fiz., **98**, 783 (1990); N. Enaki, M. Macovei, Phys. Rev. A **56**, 3274 (1997).R. Marrus and R. Schmieder, Phys. Rev. A **5**, 1160 (1972).
- [10] R. Marrus and R. Schmieder, Phys. Rev. A **5**, 1160 (1972).
- [11] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
- [12] N. Enaki, M. Macovei, Phys. Rev. A **56**, 3274 (1997).
- [13] Gross M., Haroche.S. Phys.Repp. **93**, 301-396 (1982).
- [14] Andreev A.V., Emelyanov V.I., and Ilinskii Yu.A. Cooperative Effects in Optics (IOP Publishing, Bristol, 1992).
- [15] W. Heitler, The Quantum Theory of Radiation 3-ed ed., London 1954, Sec. 8]
- [16] Y. Cheng, B. Xia 2007 arXiv: 0706.0960v2.
- [17] X. Lu, J. H. Brownell, and S. R. Hartmann, Laser Physics, **5**, 522 (1995).